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CENTRIPETAL AND/OR CENTRIFUGAL

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1. INTRODUCTION

The author considers at length the array of contradictory answers provided by university students when examined on the rotation of rigid bodies in classical mechanics during their undergraduate courses. The rotation of a body can be explained simply both in terms of kinematics and dynamics, yet this is often complicated by the various definitions formulated by different authors.

As a result some students are uncertain whether a person sitting on a merry-go-round is acted on by centripetal or centrifugal force; some students even consider that centrifugal acceleration (which is actually non-existent) is the reason for the existence of centrifugal force.

The author intends to show that the students' misunderstanding arises from two different sources: in the available literature, different forces are called by the same terms and, secondly, different terms are used to name identical forces. This Babylonian confusion of tongues has a long tradition, stretching back to Newton's Principia.

This paper tries to distinguish and compare different definitions, formulations and Terminology related to the rotation of rigid bodies. By discussing a few worked-out examples in detail the author hopes that any existing misunderstandings might be clarified.

2. Dictionary views

The adjectives *centripetal* and *centrifugal* according to the **Cambridge International Dictionary**.

- *Centripetal ... tending to move towards the point around which it is turning.*
- *Centrifugal ... tending to move away from the point around which it is turning.*

Comment: So far, so good.

Centripetal and centrifugal forces according to the **Merriam-Webster Collegiate Thesaurus**.

- *Centripetal force ... the force that is necessary to keep an object moving in a curved path and that is directed inward toward the center of rotation (a string on the end of which a stone is whirled about exerts centripetal force on the stone).*
- *Centrifugal force ... the apparent force that is felt by an object moving in a curved path that acts outwardly away from the center of rotation.*

Comment: Here, the centripetal force is defined by means of a mechanical constraint – in this case it is the force in the string. Centrifugal force is defined by means of an important and valid attribute *apparent*, however, the term is not explained in the text. I will try to clarify this later.

Centripetal and centrifugal forces according to the **Oxford Dictionary of English.**

- *Centripetal force ... a force which acts on a body moving in a circular path and is directed towards the centre around which the body is moving.*
- *Centrifugal force ... a force, arising from the body's inertia, which appears to act on a body in a circular path and is directed away from the centre around which the body is moving.*

Comment: Again, the centripetal force is defined as the constraint force. The centrifugal force is simply explained by means of inertia and thus by means of d'Alembert's formulation of Newton's second law.

Centripetal and centrifugal forces according to **Wikipedia**

- *Centripetal force* ... en.wikipedia.org/wiki/Centripetal_force

A centripetal force (from Latin centrum, "center" and petere, "to seek") is a force that makes a body follow a curved path. Its direction is always orthogonal to the motion of the body and towards the fixed point of the instantaneous center of curvature of the path. Isaac Newton described it as "a force by which bodies are drawn or impelled or in any way tend, towards a point as to a centre". In Newtonian mechanics, gravity provides the centripetal force responsible for astronomical orbits. One common example involving centripetal force is the case in which a body moves with uniform speed along a circular path. The centripetal force is directed at right angles to the motion and also along the radius towards the centre of the circular path. ... The magnitude of the centripetal force on an object of mass m moving at tangential speed v along a path with radius of curvature r is: $F = ma_c = \frac{mv^2}{r}$ *where* a_c *is the centripetal acceleration. ... In the case of an object that is swinging around on the end of a rope in a horizontal plane, the centripetal force on the object is supplied by the tension of the rope.*

Comment to the definition of centripetal force: In the text, the centripetal force is described as occurring in two particular situations. First, in the case of planetary motion, it is attributed to gravity. Second, in the case of an object moving around a fixed point by constant velocity along a circular path, it is attributed to a constrained force. This is in agreement with Newton's formulations as shown in Appendix 2. A problem arises, however, with the above formula. It does not apply to both situations and it oversimplifies the explanation, which might well cause confusion. Newton's second law, formulated for a mass point in an inertial coordinate system, states that the inertial force is equal to the sum of external forces. So we should write:

1. In the case of a planet, of mass m, orbiting the Sun, of a mass M , at the immediate distance r , we, observing the Newton's formulations and terminology (see Appendix 2), should write the equations of motion in the form $ma_c = F$,

$$F = \kappa \frac{mM}{r^2}$$

where a_c is the centripetal acceleration and $F = \kappa \frac{mM}{r^2}$ is the gravitational force, with κ being the gravitational constant. The force ma_c, in agreement with Newton's terminology, is the inertial force[2].

2. In the case of an object of mass *m*, swinging at the end of a rope of length *r* in a horizontal plane, we have $ma_c = F$ again, but *F* is the constraint force corresponding to the tension of the rope.

Thus, in both cases the force has the character of a constraint force *F* (either the gravitational force or the force in the rope), which **is equal, but not identical to, the inertial force** ma_c.

Furthermore, the final sentence of the Wikipedia definition contains a strange formulation: "*the centripetal force is* ***supplied*** *by the tension of the rope*". In mechanics we generally speak either of the equilibrium of forces or the equations of motion. A supplied force is a term which provides no useful information.

 Centrifugal force ... en.wikipedia.org/wiki/Centrifugal_force
In Newtonian mechanics, the ***centrifugal force*** *is an inertial force (also called a 'fictitious' or 'pseudo' force) directed away from the axis of rotation that appears to act on all objects when viewed in a rotating reference frame. ... The term has sometimes also been used for the force that is a reaction to a centripetal force. ... In terms of an inertial frame of reference, the centrifugal force does not exist. ... Consider a stone being whirled round on a string, in a horizontal plane. The only real force acting on the stone in the horizontal plane is the tension in the string (gravity acts vertically). There are no other forces acting on the stone so there is a net force on the stone in the horizontal plane. ... In order to keep the stone moving in a circular path, this force, known as the centripetal force, must be continuously applied to the stone. ... A body undergoing curved motion, such as circular motion, is accelerating toward a center at any particular point in time. This centripetal acceleration is provided by a centripetal force …*

Comment on the definition of centrifugal force: The statement "*The term* (meaning the centrifugal force) *has sometimes also been used for the force that is a reaction to a centripetal force*." puts together two expressions from different and disjunct worlds. The first word is one of an observer in an inertial frame of reference (situated at a fixed motionless point in the Universe), within which the term centrifugal force has no meaning. The second word is one of an observer in a non-inertial frame of reference (sitting on a planet or on a rotating stone, from the above example). The latter observer could easily write an equation stating that centrifugal force is equal to the constraint force, which might be either the gravitational force or the force in the string.

The last sentence of the above text, i.e. "*This centripetal acceleration is provided by a centripetal force*" … is rather misleading. Strictly speaking, any acceleration is "provided" by the motion around the curved trajectory. However, there is a relation between forces and accelerations but of a different origin. If a body moves with acceleration, it requires a force to cause it. Alternatively, an applied force evokes the motion of a body with acceleration. This miraculous tendency is called inertia, but from Newton's times nobody knows why this is so. See Appendix 2.

3. Blunders

Examples of terminological inaccuracies are numerous. The one from the TutorVista cited here is available at physics.tutorvista.com/motion/centrifugal-acceleration.html. The site prides itself on being

… A pioneering online tutoring company. TutorVista provides quality online tutoring to students across the world at a very affordable price. TutorVista has a diverse and highly educated tutor base of over 2000 teachers..

A few examples of errors on the site

- *If a body is moved on circular path then it has a rotational acceleration due to rotational acceleration.*

Comment: This is nonsense.

- *… The centrifugal acceleration is the acceleration that is on the curved path. When a car takes a U turn on the curved way then it does not fall towards the centre. This is due to the centrifugal acceleration, which makes the motion of the car.*

Comment: There are several problems with this definition. First of all, there is no such thing as centrifugal acceleration. Furthermore, the reader of the above text is not informed what centrifugal acceleration is, but only where it occurs. Also, students are led to draw the erroneous conclusion that it is centrifugal acceleration which is responsible for the curved motion of a car. The curved motion of a car is, in fact, evoked by constraint forces (between the wheels and the road). Due to the motion along the curved trajectory, centripetal acceleration is induced, not centrifugal. According to Newton's law the inertial force arises, which, in this case, is equal to the constraint force. The inertial force and the constraint force are two distinct quantities, which are equivalent, but not identical.

- *… Centripetal force is the force acting on the body, which makes the body move towards the center of axis of rotation. This force acts along the radius towards the center of the circle. This force is called* ***centrifugal force*** *and if we* *consider its velocity we can see that the body moving in a circular path will be having linear velocity, so, such a change in the linear velocity, taking place in this body with respect to time, is called* ***centrifugal acceleration.***

Comment: This definition is highly problematic. The change of a velocity of a point subjected to a circular motion with a constant angular velocity is a vector, which **always** (regardless of the sense of rotation) tends to the centre of rotation. This change of velocity with respect to time is called centripetal acceleration. For details see Appendix 1. It should be again emphasized that **there is no such thing as the centrifugal acceleration.**

Another, slightly more complicated, blunder

Can be found at en.wikipedia.org/wiki/Centrifugal_force. Here, a misunderstanding of the nature of the centrifugal force, the Coriolis force and gravitational force lead to erroneous conclusions being drawn related to the discussed weight of the equatorial express train described below.

Wikipedia gives the following example.

Imagine a railway line running round the Earth's equator, with a train running at high speed in the opposite direction to the Earth's rotation. The train runs at such a speed that, in an inertial (nonrotating) frame centered on the Earth, it remains stationary as the Earth spins beneath it. In this inertial frame the situation is easy to analyze. The only forces acting on the train are its gravity (downward) and the equal and opposite (upward) reaction force from the track. There is no net force on the train and it therefore remains stationary.

In a frame rotating with the Earth the train is moving in a circular orbit as it travels round the Earth. In this frame, the upward reaction force from the track and the force of gravity on the train remain the same, as they are real forces. However, in the Earth's (rotating) frame, the train is traveling in a circular path and therefore requires a centripetal (downward) force to keep it on this path. Because we are using a rotating frame, we must, as always, apply the (fictitious) centrifugal force to the train. This is equal in value to the required centripetal force but acts in an upward direction — the opposite direction to that required. It would therefore seem that there is a net upward force on the train and it should therefore accelerate upward.

In order to explain this paradox we must note that the train is in motion with respect to the rotating frame and we must therefore, in addition to the centrifugal force, add the Coriolis force. In this particular example, this acts in a downward direction and is equal in value to twice the centrifugal force thus canceling out the centrifugal force and supplying the necessary centripetal force to keep the train in its circular path.

Comment

The text to be discussed is underlined. Despite what the text above states, there is actually no paradox to explain. Instead, the interplay of centripetal and Coriolis accelerations and corresponding centrifugal and Coriolis forces need to be properly understood. In addition, the role of Newton's gravitational law has to be taken into account.

The formulation of the task then, is as follows. The train runs at high speed round the Earth's Equator in the opposite direction to the Earth's rotation and its speed is the same, but of opposite direction, as the surface point of the Earth just beneath the train.

The **outside observer** firmly connected to fixed stars, i.e. to the inertial frame of reference, sees the train as a stationary object with zero velocity and with zero acceleration. Hence, there are no inertial forces and there is no need to mention dynamics. The only forces acting on the train are its weight and the reaction force from the track. This is due to the validity of Newton's gravitational law.

The **inside observer**, sitting in the train (which is a non-inertial frame of reference) is subjected to two simultaneous motions – the relative rotation of the train with respect to the track and the carrier rotation of the track (together with the Earth) with respect to fixed stars. To each rotation a single normal (centripetal) acceleration is attributed. Furthermore, there is the Coriolis acceleration, pointing upward, due to the carrier rotation of the Earth and the velocity of the train with respect to the track. If the speed of the train is the same as that of the surface point of the Earth just beneath the train, but in the opposite direction, then the sum of two centripetal accelerations is the same, but of opposite direction, as the Coriolis acceleration. From it follows that the resulting acceleration is equal to zero.

According to d'Alembert principle, the apparent inertial forces are proportional to their respective negative accelerations. To each centripetal acceleration a single corresponding centrifugal force is attributed. Furthermore, there is the Coriolis force which is of the same magnitude but of opposite direction, than the sum of two centrifugal forces. Thus, in this particular case the sum of apparent inertial forces is equal to zero.

Of course, Newton's gravitational law is valid independently of the Earth's and train's motions, thus, as before the remaining forces are the weight of the train and the reaction force from the track.

It is rewarding that we have obtained the same results both for inertial and non-inertial observers independently of the method of observation.

In the commented Wikipedia text one of centrifugal forces is omitted, and a resulting non-equilibrium is explained incorrectly by introducing a non-existing centripetal force, without mentioning the role of gravity. Newton's gravitational force is in fact, responsible for keeping the train on the Earth's surface.

The problem of the equatorial express is analyzed in detail in Appendix 4.

4. A simple example, which might shed light on the Babylonian confusion of tongues In order to remove some of the confusion, let us examine a simple example. Consider a stone being whirled round on a string, in a horizontal plane.

The effect of gravity is neglected here. There are two alternative and equivalent ways how to tackle the problem – using either Newton's or d'Alembert's formulation. See Fig. 1.

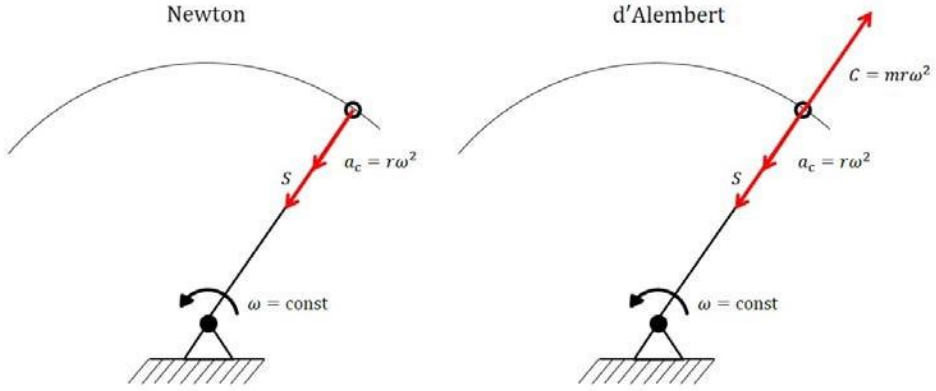

Fig. 1. Newton and d'Alembert.

Newton's formulation, i.e. the equation of motion in the form $ma_c = S$, with a_c being the centripetal acceleration, is applicable for an observer in an inertial frame of reference, for somebody who is located at a fixed point of the Universe. Newton's second law has a form of equivalence of forces. For a mass particle, it states that a product of mass and acceleration is equal to the sum of acting forces. Newton (see Appendix 2) calls the force on the left hand side of the equation, i.e., ma_c , the inertial force, while the constraint force, the force in the string, i.e. S , he denotes by the term centripetal force. We have to pull on the string to keep the stone in the circle. In this case, both forces have the same direction and the same magnitude, but they are not identical. They are of different origins.

D'Alembert's formulation is applicable for an observer in the non-inertial frame of reference, for somebody, who is located on the rotating particle. D'Alembert showed that one can write equations of motion by means of equivalent, seemingly static, equilibrium equations (sometimes erroneously called dynamic equilibrium conditions), by adding the so-called apparent inertial force. Generally, the apparent inertial force is a product of mass and negative acceleration. See [1], [2].

To correctly use Newton's law, the apparent inertial force must be added. In this case the **centrifugal force** $\vec{C} = -m\vec{a}_c$, where $a_c = |\vec{a}_c| = r\omega^2$ is the magnitude of **centripetal acceleration.** The resulting conditions of dynamic equilibrium are not the static conditions in the proper sense of the word – they are valid just for the given position and time. See [2], [3]. D´Alembert's formulation has a form of the equilibrium of forces – meaning that the sum of all forces is equal to zero. In scalar notation, where arrows in Fig. 1 indicate the direction of accelerations and forces, we might write, S-C= 0, where $C = mr\omega^2$ is the magnitude of the centrifugal force. Notice that the centrifugal force $\vec{C}$, as a vector, has an opposite sign with respect to the vector of centripetal acceleration $\vec{a}_c$. The corresponding scalar equation S- C= 0 comes from the idea of the free-body-diagram reasoning, which is based on the idea of replacing the effects of constraints by equivalent forces – in this case the string is mentally cut and replaced by an equivalent force, say S . To an observer sitting on the rotating particle the centrifugal force appears as the external force – not 'apparent' at all.

Newton's and d´Alembert's formulations, written in scalar notations with directions of forces indicated by arrows in Fig. 1, are

Newton: $ma_c = S$ and d'Alembert: S-C = 0, where $C = ma_c$, $a_c = r\omega^2$.

Equations describing the motion are seemingly identical but have a different background.

$$mr^2\,\omega = S \Leftrightarrow S - mr\omega^2 = 0.$$

To summarize briefly:

In Newton's formulation the term ma_c is the inertial force. In d'Alembert's formulation the variable C is the centrifugal force. In both formulations the term a_c is the centripetal acceleration and S is the constraint force – the force in the string.

To summarize at length:

For an observer in the inertial frame of reference, who is using Newton's formulation, the product of mass and acceleration should be called the inertial force. Calling it the centripetal force is misleading because this term is usually reserved for the constraint force. See Appendix 2. It should be emphasized that for an observer in the inertial frame of reference the term centrifugal force has no meaning.

An observer in the non-inertial frame of reference, who is using the d'Alembert's formulation, and writes dynamical equations of equilibrium, has to add apparent inertial forces to existing external forces. Apparent inertial forces are defined as a product of mass and negative acceleration. These apparent inertial forces seemingly arise out of nothing – yet they do have a sound origin based on the transformation of coordinates between the stationary (inertial) and accelerating (non-inertial) frames of reference. See Appendix 3. In our example with the rotating particle, the role of the apparent inertial force is played by the centrifugal force, while the external force is the constraint force in the string. The term *apparent* is used by stationary observers. For observers in the accelerating frame of reference the centrifugal force can be felt and could be measured and appears not apparent but quite real. This might be another source of confusion.

5. Conclusion – in the name of a gentle and futile attack on windmills

One of reasons leading to this confusion of tongues relates to the terminology describing rotation of bodies in mechanics, not only are the same terms often used for different kind of forces, sometimes different terms describe identical forces. In addition, confusion might also arise because of two possible observation points. These are either from the stationary inertial frame of reference or from the accelerating one, i.e. the non-inertial, frame of reference. Newton's and d'Alembert's formulations are proper tools corresponding to these two viewpoints and lead to identical results. An example, presented in Appendix 3, tries to clarify the matter.

Newton's formulation [4] is applicable for an observer in an inertial frame of reference. Newton's second law for a particle has the form of the equivalence of forces. The *inertial force,* product of mass and acceleration, is equal to the sum of external forces.

D'Alembert's formulation [2] is applicable for an observer in a non-inertial frame of reference, who feels the inertial effects of acceleration. It allows the person to write equations of motion by means of equivalent equilibrium equations (sometimes improperly called dynamic equilibrium conditions. These are however valid for a given position and time, only) by adding the so-called apparent inertial forces, which – together with other external forces – are in mechanical equilibrium, meaning that the sum of all forces is equal to zero. The *apparent inertial force* is a product of mass and negative acceleration. Generally, there are other apparent forces, such as Euler, centrifugal and Coriolis forces which are proportional to negative tangential, centripetal and Coriolis accelerations respectively. See Appendix 1.

To be clear and consistent, we should distinguish the terms *inertial force* and *apparent inertial force*. The ***inertial force*** **is a product of mass and acceleration**. The ***apparent inertial force*** **is a product of mass and negative acceleration**. Not many authors observe this terminological rule and in the latter case the adjective *apparent* is often dropped. Consequently, the term inertial force is used for whatever a particular author finds suitable. Compare [2] and [6].

In Newton's Principia (see Appendix 2) the term **centripetal force is reserved for the external forces, which might be of different origins** – a constraint force, gravitational force, magnetic force, etc. In this respect, most publications follow that lead, but at the same time they often claim that the centrifugal force is ma_c. According to Newton's terminology, the product of mass and acceleration ma_c is the inertial force, not the centrifugal force. **The inertial force and the centripetal (constraint) force have the same size and direction but are distinct in nature and not identical.**

There is no such a thing as centrifugal acceleration. The change of a velocity of a particle subjected to a circular motion with a constant angular velocity is a vector, which **always** (regardless of the sense of rotation) tends to the center of rotation. This vector, the change of velocity with respect to time, is called centripetal acceleration. For details see

Appendix 1.

The centripetal force should not be considered the opposite of the centrifugal force. The centripetal force, in the sense of ma_c, is a misleading term, since the product ma_c is by the definition the inertial force. Furthermore, for an observer in a stationary, inertial, frame of reference, the term centrifugal force has no meaning. The existence of centrifugal force, which can be felt and measured in the accelerating, non-inertial frame of reference, can formally be explained by means of coordinate transformation between the stationary and accelerating frames of reference.

Appendices

Appendix 1. Kinematics – Acceleration components in non-inertial frame of reference

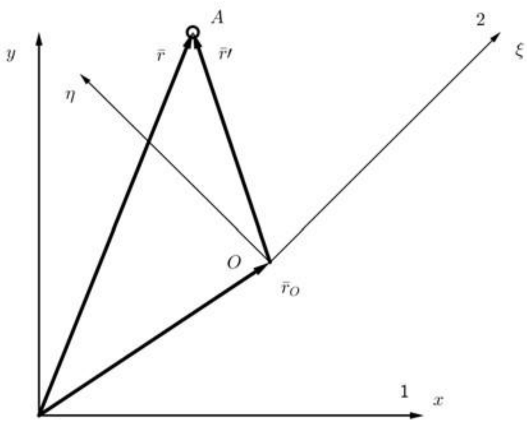

Fig. A1_1. Kinematics of relative motions.

Consider an inertial coordinate system (*x y*,), labeled 1 in Fig. A1_1. In this coordinate system there is another system (ξ,η), labeled 2. The position of the origin *O* of the system (ξ,η) is determined by the vector r_O. The origin *O* has the velocity v_{ow}. The system (ξ,η) moves with respect to the system (*x y*,) and also rotates around the origin *O* with the angular velocity $\overline{\omega}$ and angular acceleration $\overline{\varepsilon}$. The point A, representing a material particle, lying in the coordinate system (ξ,η), moves as well. Its position with respect to the coordinate system(*x y*,) is defined by the vector *r*, while the position with respect to the system (ξ,η) is defined by the vector $\overline{r}'$. The coordinate system (ξ,η) is obviously noninertial.

The time derivative of the angular velocity $\overline{\omega}$, i.e. the angular acceleration $\overline{\varepsilon}$, is independent of the choice of the coordinate system. So

$$\left[\frac{\mathrm{d}\overline{\omega}_{21}}{\mathrm{d}t}\right]_1 = \left[\frac{\mathrm{d}\overline{\omega}_{21}}{\mathrm{d}t}\right]_2 = \overline{\varepsilon}_{21} .$$

For the positional vectors we can write

$$\overline{r} = \overline{r}_O + \overline{r}' .$$

The time derivative of a positional vector *r* is defined as a vector having the direction of the trajectory of the motion of a point the positional vector is pointing to. See A1_2. In the limit

$$\lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{v} .$$

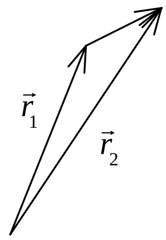

Fig. A1_2. Time derivative of a vector.

So the time derivative of a vector is the velocity of its end point.

The velocity of point A, with respect to the coordinate system 1, is given by the time derivative of the positional vector $\overline{r}$ in that system, i.e. (*x y*,), which has to be equal to the sum of time derivatives of vectors $\overline{r}_O$ a $\overline{r}'$ in the same system, thus

$$\overline{v}_A = \left[\frac{\mathrm{d}\overline{r}}{\mathrm{d}t}\right]_1 = \left[\frac{\mathrm{d}\overline{r}_O}{\mathrm{d}t}\right]_1 + \left[\frac{\mathrm{d}\overline{r}'}{\mathrm{d}t}\right]_1 =$$

… The first term is the velocity of the point O,

$$= \overline{v}_O + \left[\frac{\mathrm{d}\overline{r}'}{\mathrm{d}t}\right]_2 + \overline{\omega}_{21} \times \overline{r}' =$$

… The second term can be expressed as a time derivative

$$= \underbrace{\overline{v}_O + \overline{\omega}_{21} \times \overline{r}'}_{\overline{v}_{carrier}} + \underbrace{\left[\frac{\mathrm{d}\overline{r}'}{\mathrm{d}t}\right]_2}_{\overline{v}_r} = \overline{v}_O + \overline{\omega}_{21} \times \overline{r}' + \overline{v}_r =$$

… Reordering terms we get

$$= \bar{v}_{\text{carrier}} + \bar{v}_{\text{relative}} = \bar{v}_{\text{carrier}} + \bar{v}_{\text{r}}\,.$$

Let's define

$\bar{v}_{\text{carrier}} = \bar{v}_O + \bar{\omega}_{21} \times \bar{r}'$ – Carrier velocity and $\bar{v}_{\text{r}} = \left[\frac{\mathrm{d}\bar{r}'}{\mathrm{d}t}\right]_2$ – relative velocity.

The acceleration of point A can be derived similarly. We start with

$$\bar{v}_{\text{A}} = \bar{v}_O + \bar{v}_{\text{r}} + \bar{\omega}_{21} \times \bar{r}'.$$

Observing the rules for derivatives of products we arrive at

$$\bar{a}_{\text{A}} = \left[\frac{\mathrm{d}\bar{v}_{\text{A}}}{\mathrm{d}t}\right]_1 = \underbrace{\left[\frac{\mathrm{d}\bar{v}_O}{\mathrm{d}t}\right]_1}_{\bar{a}_O} + \underbrace{\left[\frac{\mathrm{d}\bar{v}_r}{\mathrm{d}t}\right]_1}_{\left[\frac{\mathrm{d}\bar{v}_{\text{r}}}{\mathrm{d}t}\right]_2 + \bar{\omega}_{21}\times\bar{v}_{\text{r}}} + \underbrace{\left[\frac{\mathrm{d}\bar{\omega}_{21}}{\mathrm{d}t}\right]_1 \times \bar{r}'}_{\bar{\varepsilon}_{21}\times\bar{r}'} + \underbrace{\bar{\omega}_{21} \times \left[\frac{\mathrm{d}\bar{r}'}{\mathrm{d}t}\right]_1}_{\bar{\omega}_{21}\times\left\{\left[\frac{\mathrm{d}\bar{r}'}{\mathrm{d}t}\right]_2 + \bar{\omega}_{21}\times\bar{r}'\right\}} =$$

$$= \bar{a}_O + \underbrace{\left[\frac{\mathrm{d}\bar{v}_{\text{r}}}{\mathrm{d}t}\right]_2}_{\bar{a}_r} + \bar{\omega}_{21} \times \bar{v}_{\text{r}} + \bar{\varepsilon}_{21} \times \bar{r}' + \bar{\omega}_{21} \times \left\{\underbrace{\left[\frac{\mathrm{d}\bar{r}'}{\mathrm{d}t}\right]_2}_{\bar{v}_r} + \bar{\omega}_{21} \times \bar{r}'\right\} =$$

$$= \bar{a}_O + \bar{a}_{\text{r}} + \bar{\omega}_{21} \times \bar{v}_{\text{r}} + \bar{\varepsilon}_{21} \times \bar{r}' + \bar{\omega}_{21} \times \left(\bar{v}_{\text{r}} + \bar{\omega}_{21} \times \bar{r}'\right).$$

So, the acceleration of point A is

$$a_{\text{A}} = \bar{a}_O + \underbrace{\bar{a}_r}_{\text{relative}} + \underbrace{\bar{\varepsilon}_{21} \times \bar{r}'}_{\text{tangential}} + \underbrace{\bar{\omega}_{21} \times \left(\bar{\omega}_{21} \times \bar{r}'\right)}_{\text{cenripetal}} + \underbrace{2\bar{\omega}_{21} \times \bar{v}_{\text{r}}}_{\text{Coriolis}}\,.$$

In conclusion, we have derived the acceleration components of a point, subjected to a motion in the non-inertial coordinate system. To achieve this it was not necessary to mention forces at all. Evidently, centrifugal acceleration does not exist. See [1], [3], and [8].

Appendix 2. – Here, we present a few definitions of mechanical terms from the pen of Newton and of his translators.

Examples are taken from the original Latin version of Principia [4] and from the first existing English translation [5].

My reason for using the Latin text [L] here is to enable us to find out how the mechanical terms were originally defined by Newton himself and whether they were not perhaps changed due to subsequent translations [E].

Newton's first law – if an object is left alone undisturbed, it continues to move with a constant velocity in a straight line if it was originally moving. Alternatively, it continues to stand still if it was previously just standing still. This law has the following formulation in Principia.

[L]. Def. III, p. 8.

*Materiæ **Vis insita** est potentia resistendi, qua corpus unumquodq; quantum in se est, perseverat in statu suo vel quiescendi vel movendi uniformiter in directum.*

[E]. DEFINITION III, p. 36 and a comment from [5].

*The **Vis insita,** or innate force of matter, is a power of resisting, by which everybody, as much as in it lies, endeavours to persevere in its present state, whether it be of rest, or of moving uniformly forward in a right line.*
*... This **Vis insita**, may, by a most significant name, be called **Vis inertiæ**, or force of inactivity. But a body exerts this force only, when another force, impressed upon it, endeavours to change its condition.*

Newton's second law and the inertial force, in the formulation, appearing in the English translation of [4] in [7], is

The change of motion is proportional to the motive force impressed and is made in the direction of the straight line in which that force is impressed.

Notice that the vector character of variables is observed, but the term mass – as a Coefficient of proportionality – is not mentioned. The generally known formula, i.e. $\vec{F} = m\vec{a}$, does not occur in Principia.

External force, inertial force and different kinds of centripetal force

[L]. Def. IV, p. 37.

Vis impressa est actio in corpus exercita, ad mutandum ejus statum vel quiescendi vel movendi uniformiter in directum. Consistit hæc Vis in actione sola, neq; post actionem permanet in corpore. Perseverat enim corpus in statu omni novo per solam vim inertiæ. EST autem Vis impressa diversarum originum, ut ex ictu, ex pressione, ex Vis centripeta.

[E]. DEFINITION IV, p. 9.

An impressed force is an action exerted upon a body, in order to change its state, either of rest, or of moving uniformly forward in a right line. This force consists in the action only; and remains no longer in the body, when the action is over. For a body maintains every new state it acquires, by its Vis inertiæ only. Impressed forces are of different origins as from percussion, from pressure, from centripetal force.

Centripetal force

[L]. Def. V, p. 9.

Vis centripeta est qua corpus versus punctum aliquod tanquam ad centrum trahitur, impellitur, vel utcunq; tendit.

Hujus generis est gravitas, qua corpus tendit ad centrum Terræ: Vis magnetica, qua ferrum petit centrum Magnetis, et vis illa, quæcunq; sit, qua Planetæ perpetuo retrahuntur a motibus rectilineis, et in lineis curvis revolvi coguntur. EST autem Vis centripetæ quantitas trium generum, absoluta, acceleratrix et motrix.

[E]. DEFINITION V, p. 37 and a comment from [5].

A centripetal force is that by which bodies are drawn or impelled, or any way tend, towards a point as to a centre.

Of this sort is gravity, by which bodies tend to the centre of the earth magnetism, by which iron tends to the loadstone; and that force, whatever it is, by which the planets are perpetually drawn aside from the rectilinear motions, which otherwise they would pursue, and made to revolve in curvilinear orbits.

... A stone, whirled about in a sling, endeavours to recede from the hand that turns it; and by that endeavour, distends the sling, and that with so much the greater force, as it is revolved with the greater velocity, and as soon as ever it is let go, flies away. That force which opposes itself to this endeavour, and by which the sling perpetually draws back the stone towards the hand, and retains it in its orbit, because it is directed to the hand as the centre of the orbit, I call the centripetal force. And the same thing is to be understood of all bodies, revolved in any orbits. ...

Appendix 3. – The same phenomenon as viewed by inertial and non-inertial observers.

Newton's law, in its simple form, i.e. $\vec{F} = m\vec{a}$, is only applicable to a particle in a so called *inertial frame of reference*. In older text books, the term inertial frame reference was nicely defined as a system which is attached to fixed stars. Such a system can be either absolutely still or moving with a constant velocity with respect to fixed stars. *A non-inertial frame of reference* is a frame which undergoes acceleration with respect to an inertial frame. Since the Universe is expanding and constantly accelerating, there are no fixed stars available and generally, no inertial frame of reference exists.

Nevertheless the Earth can be – for many (but not all) engineering applications – approximately considered as the inertial frame of reference since its orbital accelerations, due to Earth's daily and annual rotations are small.

Consider a simple task where the Earth is considered as the inertial frame of reference while the street car, accelerating on tracks laid on the Earth's surface, serves as an example of a non-inertial frame.

Let the coordinate system x, y represent our approximate inertial reference system, firmly connected to the Earth. Our task is to analyze the trajectory of a particle having the mass m , being propelled by a constant force F (imagine a small rocket engine attached to the particle), which resides in the street car moving in x direction with constant acceleration a along a straight horizontal track, while the non-inertial system of reference, i.e. x', y' is firmly connected to the accelerating street car. See Fig. A3_1. The initial velocities of the particle with respect to the street car are known. Only the planar motion is considered and the Earth's gravity is taken into account.

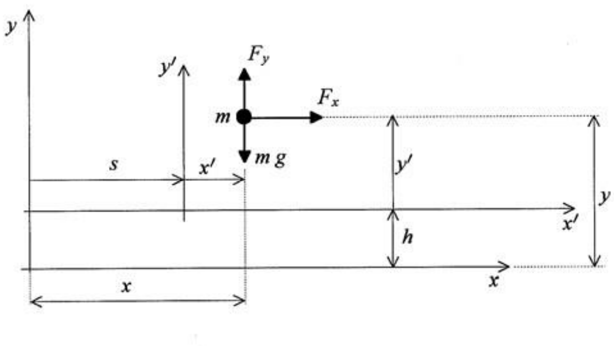

Fig. A3_1. A particle in an accelerating street car.

Using Newton's formulation, the equations of motion, relative to the Earth, are

$$m\ddot{x} = F\cos\alpha, \quad m\ddot{y} = F\sin\alpha - mg. \qquad \text{(A3\_1)}$$

$$\ddot{x} = \frac{F}{m}\cos\alpha, \quad \ddot{y} = \frac{F}{m}\sin\alpha - g.$$

Initial conditions of the street car.

At time $t = 0$ the axes y and y' coincide, while there is a constant distance h between the axes x' and x . The street car's initial velocity is zero.

Initial conditions of the particle.

At the beginning, the particle resides at the origin of x', y' system and its initial velocity components are $v_{x0} = v_{x'0}$, $v_{y0} = v_{y'0}$.

Taking into account the prescribed initial conditions, the double integration of Eqs. (A3_1), with respect to time, gives the particle velocities and the particle coordinates as functions of time, as seen by an outside observer.

$$v_x = v_{x0} + \frac{Ft}{m}\cos\alpha, \quad v_y = v_{y0} + \frac{Ft}{m}\sin\alpha - gt. \qquad \text{(A3\_2a)}$$

$$x = v_{x0}t + \frac{Ft^2}{2m}\cos\alpha, \quad y = v_{y0}t + \frac{Ft^2}{2m}\sin\alpha - \frac{1}{2}gt^2. \qquad \text{(A3\_2b)}$$

Due to the prescribed constant acceleration of the street car, the transformation of the coordinates, between the Earth coordinate system and the street car's coordinate system, is as follows.

$$x = x' + s, \quad s = \frac{1}{2}at^2, \quad y = y' + h, \quad h = const. \qquad \text{(A3\_3)}$$

The velocities $v_{x'0}$, $v_{y'0}$ belong to the particle. Hence

$$\ddot{x} = \ddot{x}' + \ddot{s}, \quad \ddot{s} = a.$$

$$\ddot{x} = \ddot{x}' + a, \quad \ddot{y} = \ddot{y}'. \qquad \text{(A3\_4)}$$

Substituting the last relation of Eqs. (A3_4) into Eqs. (A3_1) gives the equations of motion of the particle relative to the accelerating street car

$$m\ddot{x}' = F\cos\alpha - ma, \quad m\ddot{y}' = F\sin\alpha - mg. \qquad \text{(A3\_5)}$$

The equations (A3_5) have the form of the *equivalence of forces*. The left hand side force (the inertial force) is equal to the sum of right hand side, i.e. external, forces.

Using the d'Alembert's principle we might write the equations of motion in an alternative form

. (A3_6)

Now, the equations of motion (A3_6) are expressed in the form of an *equilibrium of forces*. The sum of all forces is equal to zero. Of course, it is not the proper "static" equilibrium; it is a sort of virtual equilibrium, expressed for a moment frozen in time.

An additional force i.e. -*ma* appears on the right hand side of equations of motion, in Eq. (A3_6), written for the non-inertial frame of reference. Cornelius Lancozs [2], calls it an *apparent force* or *d'Alembert force*, which – for an observer attached to the Earth – seemingly emerges out of nothing. Evidently, it is the acceleration of the moving frame of reference which is responsible for the existence of that force.

This force might be considered fictitious only for outside observers, who are firmly standing on the Earth and build up their reasoning without knowing that the particle is in the accelerating street car, which leads them to Eq. (A3_1).

For the inside observer that force – being often paradoxically called *fictitious* – is almost *real* since it could be physically felt and experimentally measured. So the currently used term, i.e. *fictitious*, might appear rather misleading to observers living in a non-inertial frame of reference, i.e. in the accelerating street car.

Such contradictory terminology appears frequently. For example, Dare A. Wells in [6] states that *we shall, throughout the book, refer to the product (mass)* $\times$ *(acceleration) as an "inertial force"*, while for C. Lancozs in [2] the inertial force is *-ma*. These two authors, as well as many others, are using the same term, i.e. inertial force, for two forces of the same magnitude differing, however, by a plus or minus sign.

Double integration of Eqs. (A3_6) with respect time gives the velocity and coordinate distribution as functions of time with respect the accelerating street car – the distributions seen by an inside observer.

$$v_{x'} = v_{x'0} + \frac{Ft}{m}\cos\alpha - at, \quad v_{y'} = v_{y'0} + \frac{Ft}{m}\sin\alpha - gt\,. \qquad \text{(A3\_7a)}$$

$$x' = v_{x'0}t + \frac{Ft^2}{2m}\cos\alpha - \frac{1}{2}at^2, \quad y' = v_{y'0}t + \frac{Ft^2}{2m}\sin\alpha - \frac{1}{2}gt^2\,. \qquad \text{(A3\_7b)}$$

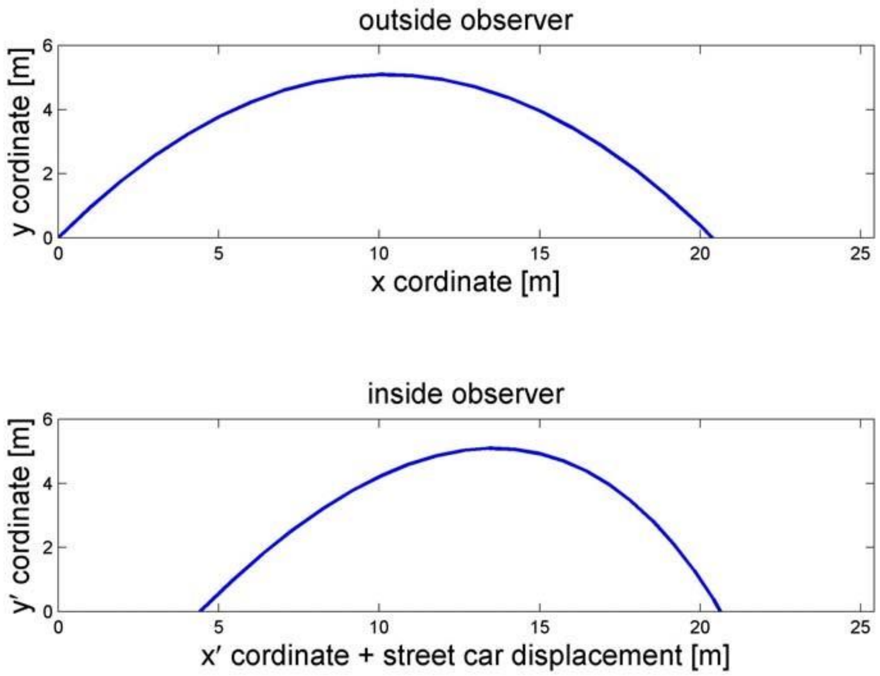

Fig. A3_2. The trajectory of a particle as viewed by two observers.

In the upper subplot of Fig. A3_2 there is a trajectory (see Eq. (A3_2b)) of the particle, fired from the accelerating street car with prescribed initial velocities, as seen by an outside observer. In order to compare the results with those of a simple oblique shot case, the value of the rocket force *F* was temporarily set to zero. The data in the lower subplot show the trajectory as seen by the inside observer. The curve, plotted according to Eq. (A3_7b), is intentionally shifted to the right by the distance the car travelled during period of the time being considered, so that one can see that the coordinates of the "hit" point, are identical for both observers. Of course, the time that elapses before the time to "hit" is identical for both observers too.

Appendix 4. – An equatorial express

Imagine a train running on the track with a constant speed around the equator of the Earth in the opposite direction to the Earth's rotation, i.e. clockwise. A simplified sketch is in Fig. A4_1. The train, represented by a sleeve, is denoted by the number 3. The equatorial track, firmly connected to the Earth, depicted as a part of the circle, is denoted by the number 2. The number 1 is a fixed point in the Universe – an inertial frame of reference.

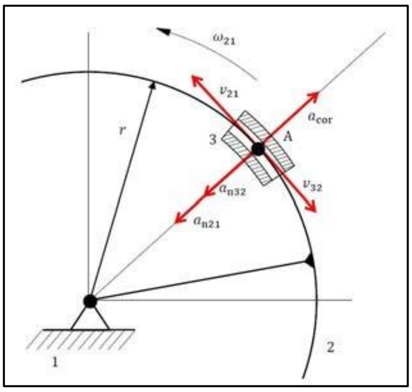

Fig. A4_1. Velocities and accelerations

Given data The radius of the Earth is r , the speed of the train with respect to the track is $v_{32} = |\vec{v}_{32}|$, the angular velocity of the Earth for its counterclockwise rotation is defined by a vector of angular velocity $\vec{\omega}_{21}$ pointing up vertically out of the picture plane. Its scalar magnitude is $\omega_{21} = |\vec{\omega}_{21}|$. The corresponding surface speed of a point on the equator, say A, is $v_{21} = r\omega_{21}$. The situation is schematically depicted in Fig. A4_1, where the directions of the speeds are indicated as well. The mass of the train *m* is concentrated at the point A.

The task

Evaluate the force reaction between the track and the train – i.e. the actual weight of the train – as a function of its relative speed with respect to the Earth.

Simplifying assumptions

Since the relative differences in the actual weight of the train, depending on its location and speed, are of the order of a fraction of one percent, the precise simplifying assumptions have to be carefully listed.

- The Earth is assumed to be a perfect sphere with a constant radius; r = 6378 km.
- The gravitational acceleration on the Earth's surface is constant and equal to g = 9.81m/s^2.
- The Earth's angular velocity is approximated by $\omega = \dfrac{2\pi}{24 \times 60 \times 60}$ 1/s.
- The orbit of Earth around the Sun is disregarded.
- The actual weight of an object depends on its location on the Earth. When weighing the object of mass *m* by a spring balance on the pole, the balance shows the value of *mg.* Using the same spring balance and the same object at the equator the weight is diminished (due to the Earth rotation) by the value of the centrifugal force, i.e. by $mr\omega^2$. The relative difference of these two values is 0.0034. The same object, measured at the equator would be lighter. Its actual weight would be 0.9966 *mg.*
- The actual weight also depends on its velocity with respect to the surface of the Earth.
- Resistance and friction phenomena are not considered.
- In Newtonian mechanics the mass of an object is considered independent of its speed.

Kinematics

The train moves with respect to the Earth, which simultaneously rotates underneath the train. In this particular case, the decomposition of motions [9] could be expressed by a symbolic notation in the form

$$31 = 32 + 21. \qquad \text{(A4\_1)}$$

This means that
The absolute motion of the train 3 with respect to inertial frame 1 is composed of

- the relative motion of the train 3 with respect the track 2 plus
- The motion of the track 2 with respect to the inertial frame 1.

For velocities we can write

$$\vec{v}_{31} = \vec{v}_{32} + \vec{v}_{21}. \qquad \text{(A4\_2)}$$

The speed (the magnitude of velocity) of the train $v_{32} = |\vec{v}_{32}|$ is constant and known. The speed of the surface point just below the train is constant as well

$$v_{21} = |\vec{v}_{21}| = \omega_{21} r \text{, where } \omega_{21} = |\vec{\omega}_{21}|. \qquad \text{(A4\_3)}$$

The resulting acceleration [9] with respect to the inertial frame is expressed by

$$\vec{a}_{31} = \vec{a}_{32} + \vec{a}_{21} + \vec{a}_{cor}. \qquad \text{(A4\_4)}$$

Generally, the acceleration vectors a_{32} and a_{21} have both tangential and normal components. In our case, both the rotation of the Earth and the velocity of the train with respect to the track are considered constant, so the tangential components of these accelerations are identically equal to zero, i.e.

$$a_{32t} = a_{21t} = 0. \qquad \text{(A4\_5)}$$

What remains are normal (centripetal) components of accelerations. Their magnitudes are

$$a_{32n} = v_{32}^2/r \text{ and } a_{21n} = v_{21}^2/r \qquad \text{(A4\_6)}$$

While their directions are indicated by arrows in Fig. A4_1.

The Coriolis acceleration is defined as a vector product of the angular velocity of rotation and the relative velocity. In our case

$$\vec{a}_{cor} = 2\vec{\omega}_{21} \times \vec{v}_{32} . \quad \text{(A4\_7)}$$

Since the vectors $\vec{\omega}_{21}$ and $\vec{v}_{32}$ are perpendicular, the magnitude of the resulting vector $|\vec{a}_{cor}|$ is

$$a_{cor} = 2\omega_{21} v_{32} = 2v_{21}v_{32} / r . \quad \text{(A4\_8)}$$

For the accepted clockwise train rotation the vector a_{cor} points out of the centre of rotation as indicated in Fig. A4_1. In this case, it is in the opposite direction with respect to directions of normal acceleration vectors a_{32n} and a_{21n}. The vector a_{cor} would have the same magnitude but an opposite direction if the train rotation were considered counterclockwise.

So the magnitude of the resulting radial acceleration with respect to the inertial frame is

$$a_{31} = mv_{32}^2 / r + mv_{21}^2 / r - 2mv_{32}v_{21} / r . \quad \text{(A4\_9)}$$

Evaluating Eq. (A4_9), for the train speed v_{32} varying from zero to $2v_{21}$, we obtain the resulting normal acceleration of the train as a function of its speed. Its normalized value, related to the gravitational acceleration, i.e. a_{31} / g, as a function of the normalized relative speed $| v_{32} / v_{21} |$, is plotted on the left hand side of Fig. A4_3.

Dynamics

The forces acting upon the mass particle, act along a single line connecting the center of rotation and the point A, at which the mass particle, representing the train, is located. See Fig. A4_2. Hence, in the subsequent analysis it suffices to express the equilibrium of forces in a Scalar form.

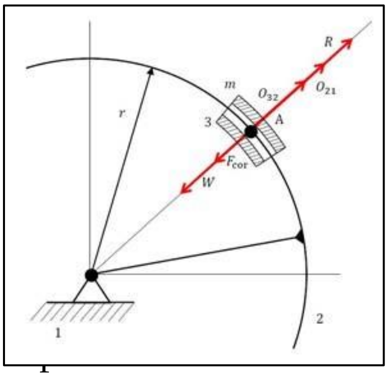

Fig. A4_2. Forces

Sitting in the train, we write the equations of motion using the d'Alembert approach [2] requiring us to consider the apparent inertia forces. Each apparent inertial force is defined as a product of the mass and the appropriate negative accelerations as follows

$O_{32} = mv^2_{32} / r$ – centrifugal force due to the relative motion 32 with centripetal acceleration a_{32n},

$O_{21} = mv^2_{12} / r$ – centrifugal force due to the carrier rotation 21 with centripetal acceleration a_{21n},

$F_{cor} = 2mv\ v32\ 21 / r$ – Coriolis force due to the carrier rotation ω_{21} and the relative speed $v32$.

Furthermore, due to Newton's gravitational law, there are the reaction force R, between the track and the train, and the train's actual weight W, which also has to be taken into account. Forces and their directions are shown in Fig. A4_2.

The equation of motion of the train (considered as a particle) seemingly has a form of "static" equilibrium

$$R + O_{32} + O_{21} - W - F_{cor} = 0 . \quad \text{(A4\_10)}$$

So the reaction force is

$$R = W + F_C - O_{32} - O_{21} = mg + 2mv_{32}v_{21} / r - mv_{32}^2 / r - mv_{21}^2 / r . \quad \text{(A4\_11)}$$

The reaction force between the track and the trains corresponds to the train's actual weight. Its relative value, i.e. $R\ mg/$, as a function of the absolute value of the relative speed $| v_{32} / v_{21} |$, is plotted in the right hand side of Fig. A4_3.

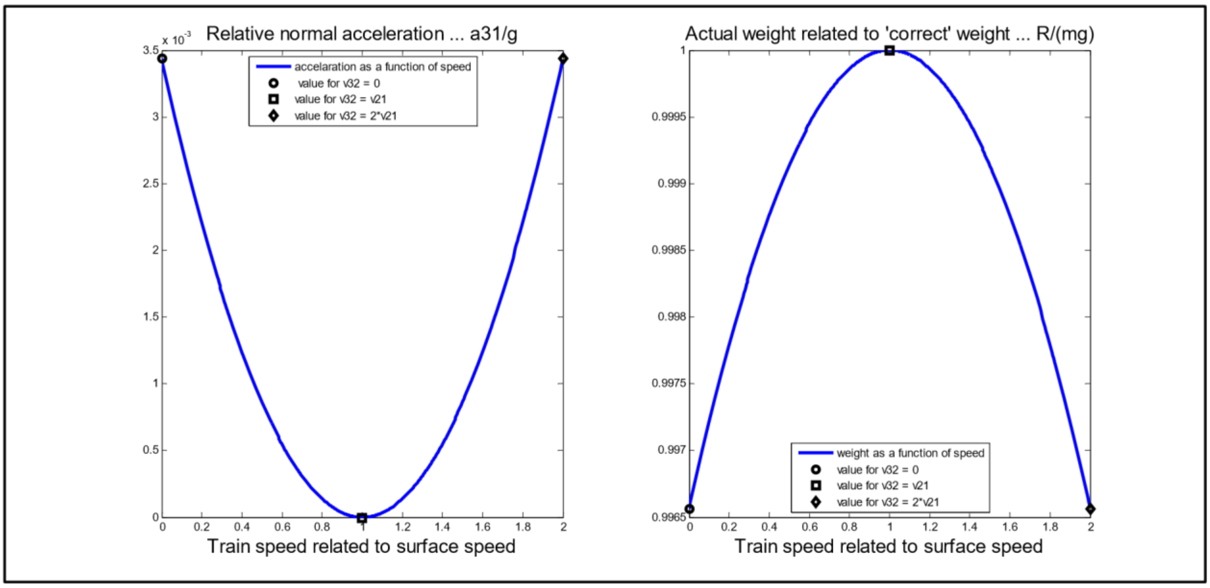

Fig. A4_3. The normal acceleration and the actual weight of a train circling the equator clockwise as functions of relative speed.

Now, a few singular cases are discussed in detail.

Case 1 – stationary train at the pole, i.e. $V_{32} = V_{21} = 0$.

If a stationary object (train) is weighed at the Earth's pole using a spring balance we would get the value of its weight (the force of gravity) which is influenced neither by the Earth's rotation, nor by the object's speed. Under these conditions the reaction between the object and the Earth is $R= mg$. This location might serve for the definition of the value of the "correct" weight.

Case 2 – stationary train at the equator, i.e. $V_{32} = 0$.

The train is stationary with respect to the Earth, so $V_{32} = 0$. In this case the "correct" weight of the train is diminished by the centrifugal force $O_{21} = mV_{21}^2/r$ due to the rotation of the Earth. Thus, the actual value of the weight is $R= mg- mr\omega^2$.

Case 3 – the train circling the equator clockwise with $\vec{v}_{32} = -\vec{v}_{21}$

The train runs on the track round the equator in the opposite direction to the Earth's rotation. For velocity vectors we have $\vec{v}_{32} = -\vec{v}_{21}$. Their magnitudes, called speeds, are identical, i.e. $v = V_{32} = V_{21}$.

The resulting velocity of the train $\vec{v}_{31}$, with respect to fixed stars, is identically equal to Zero, which directly comes from Eq. (A4_2).

The resulting acceleration $\vec{a}_{31}$, with respect to fixed stars, according to the rearranged Eq. (A4_9), is

$$a_{31} = mv^2/r + mv^2/r - 2mvv/r = 0$$

And is equal to zero as well.

So, the **outside observer**, firmly attached to the fixed stars, i.e. to the inertial frame of reference, sees the train as a stationary object with zero velocity $\vec{v}_{31}$ and with zero acceleration $\vec{a}_{31}$.

In the inertial frame of reference, the train is stationary and is subjected to no acceleration. As a result, there are no inertial forces and there is no need to talk about dynamics. The only forces acting on the train are the reaction force R between the track and the train and its weight W resulting from Newton's gravitational law. Applying the static conditions of equilibrium leads to $R= mg$. So, in this case the actual weight of the train, circling the equator clockwise with $\vec{v}_{32} = -\vec{v}_{21}$, is the same as that measured on the pole.

What about the **inside observer**, travelling on the equatorial train? Of course, he/she uses the same equation (A4_4) as far as the acceleration is concerned, but his/her attention is concentrated on the right hand side of the equation. The resulting zero on the left hand side is composed of three non-zero components. And according to d'Alembert's principle each acceleration component is complemented by a corresponding apparent (fictitious) inertial force, defined as a product of mass and negative acceleration, in agreement with Eq. (A4_11). So in this case we have

$$R = W + F_{cor} - O_{32} - O_{21} = mg + 2mv^2/r - mv^2/r - mv^2/r\,.$$

The Coriolis force and the two centrifugal forces cancel each other out and thus the reaction force is $R= mg$.

We have obtained the same result both for inertial and non-inertial observers independently of the method of observation. This is rewarding.

Finally, what is the speed V_{32} of the train with respect to the Earth satisfying the conditions of Case 3? As far as the magnitudes of vectors are concerned we have

$$v = v_{32} = v_{21} = r\omega_{21} = 6378000\frac{2\pi}{24\times 60\times 60} = 463.8 \text{ m/s} = 1670 \text{ km/h} .$$

This is a high value, but not excessively so. At the expense of this relatively high relative speed, needed for satisfying Case 3 conditions, we get the same weight as that measured on the pole. With respect to the weight of the stationary train (Case 2, $v_{32} = 0$), the train traveling clockwise, with $\vec{v}_{32} = -\vec{v}_{21}$, is heavier by 0.34 %.

Case 4 – train circling the equator clockwise with $\vec{v}_{32} = -2\vec{v}_{21}$.

Both the acceleration and the reaction force are the same as in the Case 2. See Fig. A4_3. The detailed analysis of this case is left to the reader.

Summary of the cases

Case	location	$\lvert\vec{v}_{32}\rvert/\lvert\vec{v}_{21}\rvert$	relative acceleration a_{31}/g	relative reaction R/mg
1	pole	--	0	1
2	equator	0	0.0034	0.9966
3	equator	1	0	1
4	equator	2	0.0034	0.9966

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